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Boundary Layer Flow in a Thin Sheet of Viscous Liquid

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SUMMARY

This paper investigates the steady flow of a thin sheet of viscous liquid down a variable incline; the Reynolds number *Re* and the Froude number *Fr* are both large, with $Fr/Re = 0(1)$. The governing equations are of the boundary layer type, and the Karman–Polhausen method is utilised to describe possible motions. It is found, when the incline becomes increasingly horizontal, that there can exist a point of separation on the bed of the stream; this is accompanied by a rapid rise in the depth of the liquid. The physical explanation is equivalent to that of classical boundary layer theory, except that the role of the pressure gradient in the external flow as the driving force is replaced by that of the force due to gravity.

1. Introduction

The flow of a liquid down an open channel is one of the basic problems in the theory of mathematical hydraulics. It has been difficult to take into account the effects of viscosity directly because of the mathematical complications. However, much success has been gained by replacing the viscous forces in the Navier–Stokes equations with a frictional resistance term derived from an empirical formula due to Manning; see, for example, Stoker [1]. A theory that avoids the necessity of using the Manning formula has recently been presented for flows with a small Reynolds number by Mei [2], and extended by Smith [3]. The behaviour could be classed as a Stokes flow. In the present paper we consider motions with a high Reynolds number that are essentially of the boundary layer type; specifically, we investigate the steady flow of a thin sheet of liquid down a variable incline.

We require that the flow within the thin sheet of liquid represents a balance between the viscous, inertial and gravitational forces. Consequently, section 2, the Reynolds number *Re* and the Froude number *Fr* are large such that $Fr/Re = 0(1)$; *Re* is effectively defined as the ratio of the volumetric flux to the kinematic viscosity. The theory developed is very similar, in fact, to the classical boundary layer theory, and the governing equations are equivalent to the boundary layer equations with the one alteration that the driving mechanism is provided by gravity rather than a pressure gradient in the external flow. As a result, many of the ideas from boundary layer theory can be utilised in this work.

For one particular bed profile, similarity variables can be defined and an exact solution of the governing equations is calculated; this is presented in section 3. Otherwise, it is necessary to use approximate methods, and the most fruitful is through an adaptation of the Karman–Polhausen technique that utilises the momentum integral (*cf.* Rosenhead [4]); this is developed in section 4. A pair of coupled first order ordinary differential equations are found for the profile of the free surface and the velocity on that free surface in terms of the slope of the bed. Numerical solutions are calculated that give the profile of the free surface in particular cases. However, in certain cases when the bed becomes almost horizontal, the numerical procedure breaks down, and all evidence points to the presence of a point of separation. Physically, the gravitational force becomes too weak to be capable of maintaining the flow, and the drag on the bed of the stream tends to zero. Beyond the point of separation there is a rapid increase in the depth of the liquid. With regard to the location of the separation point, the Karman–Polhausen profile is known to correctly predict its existence, although somewhat underestimating the exact location. Because one of the main purposes of this paper is just to display the phenomenon, it has not been felt necessary to adapt some of the more refined techniques of boundary layer theory to determine the exact position.

It must be emphasized that there is a distinct difference between this separation and the hydraulic jump phenomenon. Watson [5] gives a clear description of the creation of the latter downstream from a sluice gate when the flow is along a horizontal plane. His results are seen (section 2) to be immediately applicable when the flow from the gate is down an incline of variable slope. It is necessary that the Froude number is large such that $Fr/Re \gg 1$ in the motion prior to the jump; beyond the jump there is a reduction in the order of magnitude of Fr. Also, to the first order, the distance from the sluice gate to the jump is independent of the angle the slope makes with the horizontal. It is when the flow is down an incline with increasingly small slope, and with $Fr/Re = 0(1)$, that the point of separation has been noted.

Flow with $Fr = 0(1)$ and $Re \gg 1$ exist only when the slope of the channel is zero. It is in this domain, in fact, that we can consider the effect of viscosity on the classical "long wave in shallow water" theory, where the liquid is assumed to be inviscid. The recent paper by Wen [8] begins this particular investigation. The present work, therefore, considers the flow regime where the Froude number lies between the value taken by Watson and that taken by Wen. Ackerberg [9] develops an analysis with $Fr/Re = 0(1)$ as he considers the formation of the boundary layer along a vertical plate when the initial flow is in free fall. (It must be noted that Ackerberg defines his Froude number from the initial data, and that one of his conclusions is that the Froude number is rapidly increased by an order of magnitude to that defined here as the flow "forgets" the manner in which it started.) Consequently, the investigation of this paper can be seen as a natural continuation of the work of Ackerberg, where the boundary layer is fully developed and the slope of the bed begins to vary from that of a uniform slope.

Now, the present investigation was prompted by observations made by the author of certain flows in a stream, as well as by a desire to further understand the effect of gravitational forces on a boundary layer in extending the work of Ackerberg. However, it is still necessary to consider the physical existence of the phenomenon of separation; the other features described (e.g. when the incline becomes steeper so the depth of the liquid decreases) do not raise this question. To this end we observe that the angle at which separation is given by the numerical integration is small in all cases; to be precise, the angle is never found to be greater than 0.06 radians (about 3°). The angle is of this order even when the data is extreme in the sense that it strains the assumption that the rate of change of the slope must be of finite order (or less). This fact in itself implies that the effect cannot be a common occurrence, for the flow will naturally break down when the slope becomes level.

Another point to consider concerns the stability of flows down an inclined plane; as with all boundary layer studies there is the uncertainty as to the possible break down of laminar flow. The calculations and experiments of Benjamin [6] and Binnie[7] do show that instabilities can develop for comparatively small Reynolds numbers, even though they are not always easy to observe. Therefore caution is required in assuming the physical existence of separation with a laminar flow; separation can take place with a turbulent boundary layer, though it is delayed considerably by turbulence.

The strongest assertions that can be made are when the analogy with the established results of boundary layer theory is noted. The governing differential equation represents exactly the same balance of forces except that the driving force of a pressure gradient is replaced by that due to gravity. It is, therefore, most reasonable to anticipate that a reduction in the gravitational force can be the direct cause of separation, in the same way that the reduction in the pressure gradient is known to give rise to separation in the flow past bluff bodies (e.g. the point of separation for the uniform flow past a circular cylinder is approximately 81° from the forward stagnation point, Rosenhead [4, p. 264, 285]). In conclusion, although the analogy with classical boundary layer gives strong evidence for the existence of the separation phenomenon, careful experiments are required for a more definitive answer to this question.

2. Formulation

The purpose of the following investigation is to consider the steady flow under gravity of a thin sheet of viscous liquid down a slope with varying incline at high Reynolds numbers; the terms thin and large are now given a precise formulation. Let $\bar{x}$ measure the distance along the bed of the incline from a fixed origin O, and $\bar{y}$ the perpendicular distance from the bed of a field point $P(\bar{x}, \bar{y})$; the free surface profile is $\bar{y} = \bar{\eta}(\bar{x})$. The bed makes the variable angle $\theta(\bar{x})$ with the horizontal. At P the velocity of the liquid in the $\bar{x}$, $\bar{y}$ directions is $\bar{u}$, $\bar{v}$ respectively, and the pressure is $\bar{p}$.

We now introduce the non-dimensional variables x, y, η, u, v, p by $\bar{x} = lx$, $\bar{y} = dy$, $\bar{\eta} = dn$, $\bar{u} = Uu$, $\bar{v} = Vv$, $\bar{p} = \rho U^2 p$ where ρ is the density of the liquid. The constants l, d are the length scales, and U, V the velocity scales in directions parallel and perpendicular to the bed profile respectively. When we write $d/l = \varepsilon$, and set $\varepsilon \ll 1$, the investigation is restricted to flows within a thin sheet of the liquid. [When l is taken to represent distances within which the slope θ has variations of finite order, it is also necessary that the magnitude of $d\theta/dx$ is no greater than $0(1)$.] However, it is now necessary to put $V/U = \varepsilon$, for only in this manner can the terms in the continuity equation, that ensures the conservation of mass, have the same magnitude. Alternatively, the defining relation for V can be seen to be $V = Ud/l$. In the non-dimensional variables, the Navier–Stokes equations now become

$$u_x + v_y = 0\,, \tag{1}$$

$$uu_x + vu_y = -p_x + (\varepsilon Fr)^{-1} \sin\theta + \varepsilon Re^{-1} u_{xx} + (\varepsilon Re)^{-1} u_{yy}\,, \tag{2}$$

$$\varepsilon^2 (uv_x + vv_y) = -p_y - Fr^{-1} \cos\theta + \varepsilon^3 Re^{-1} v_{xx} + \varepsilon Re^{-1} v_{yy}\,, \tag{3}$$

where $Re = Ud/\nu$ is the Reynolds number, $Fr = U^2/gd$ is the Froude number; g is the acceleration due to gravity and ν the (constant) coefficient of viscosity. The boundary conditions, which require zero velocity on the bed, and zero stress together with no normal flux on the free surface, are stated later after the basic approximations have been completed. We do note now that the effect of surface tension is not included.

There is an exact solution to the equations (1)–(3) when the bed has constant slope α; this is given by

$$u = Re\, Fr^{-1} \sin\alpha\,(hy - \tfrac{1}{2}y^2), \quad v = 0, \quad p = Fr^{-1} \cos\alpha\,(h - y)\,, \tag{4}$$

where h is the constant (non-dimensional) depth.

When there is a balance between the viscous, inertial and gravitational forces in a flow at large Reynolds numbers, it is clear from (2), (3) that we set $Fr = \varepsilon^{-1}$, $Re = \varepsilon^{-1}$. Consequently, the pressure is $0(\varepsilon)$ and can be neglected, so there remain the equations

$$u_x + v_y = 0 \tag{5}$$

$$uu_x + vu_y = S(x) + u_{yy}\,, \tag{6}$$

where $S(x) = \sin\theta$. The system (5), (6) is equivalent to the boundary layer equations with the effect of a pressure gradient in the external field replaced by that of a gravitational force due to the slope of the bed. These equations will be considered in greater detail in the later sections within the context of boundary layer theory, though we note in passing that they are similar to the pair investigated by Ackerberg [8] in the particular case $S(x) = 1$.

When the Froude number satisfies $Fr \gg \varepsilon^{-1}$ the gravitational forces can be neglected, so that the flow represents a balance between the viscous and inertial forces only; the resultant equations are the same as (5), (6) except that the term $S(x)$ is absent. These are just the equations solved by Watson [5] in the particular case $S(x) \equiv 0$, and so his results are immediately applicable. Therefore, there will be a linear increase in the surface elevation from just beyond the commencement of the flow (at a sluice gate for example) until the occurrence of a hydraulic jump. To the first order the distance between these positions is independent of the slope of the incline, because of the negligible influence of the gravitational forces. Beyond the jump, the effect of gravity becomes equal in magnitude to the viscous and inertial forces, and further considerations are based on equations (5), (6). There is, therefore, a change in the order of magnitude of the Froude number across the jump. It can be deduced from the calculations of Watson that $Fr_0/Fr_1=(d_1/d_0)^3$ when the zero and unity subscripts represent values before the beyond the jump respectively. (The length scale l is taken to be the distance from the commencement of the flow to the hydraulic jump). The conservation of mass is the basic law assumed for this result.

If the pressure is to have a non-negligible effect on the flow, it is clear from both (3) and (4) that $Fr=0(1)$. The pressure is then hydrostatic and given by $p=\cos\theta(\eta-y)$.

However, the gravitational term $(\varepsilon Fr)^{-1}\sin\theta$ in (2) then dominates all other terms unless $\theta(x)\equiv 0$; this is the situation discussed by Wen [9]. For variable θ, there is no steady flow within which the pressure plays a dominant role.

3. An Exact Solution

In the rest of this paper, the equations (5), (6) are considered. The necessary boundary conditions are derived from taking the approximation $Fr=Re=\varepsilon^{-1}$, $\varepsilon\ll 1$ in the most general form as given by Wehausen and Laitone [10], for example. The conditions are seen to be

$$u=v=0 \text{ on } y=0\,, \tag{7}$$

$$u_y=0 \quad\text{and}\quad u\eta_x=v \text{ on } y=\eta(x)\,, \tag{8}$$

together with

$$\int_0^\eta u\,dy=\text{constant}=1\,. \tag{9}$$

Assume that there exists a similarity solution of the form

$$u(x,y)=U(x)f(\zeta)\,,\qquad \zeta=y/\eta(x) \tag{10}$$

where U is the velocity on the free surface. Mass conservation, (9), implies that

$$U\eta=\text{constant}\,. \tag{11}$$

The equation of continuity, (5), then indicates $v=U\eta'\zeta f(\zeta)$, and substitution into the momentum equation (6) further requires

$$S\eta^3=\text{constant} \quad\text{and}\quad U'\eta^2=\text{constant}\,. \tag{12, 13}$$

The analysis is similar to that of Watson [5], and is not reproduced here. It is necessary that, given S, there exist functions U and η that satisfy (11)–(13); the only such representations are seen to be

$$S(x)=\mu(x+x_0)^{-3}\,,\quad \eta(x)=\omega(x+x_0)\,,\quad U(x)=\lambda(x+x_0)^{-1} \tag{14}$$

for positive constants μ, ω, λ and all values of x_0. The function $f(\zeta)$ satisfies the non-linear ordinary differential equation $f''+\mu\omega^2\lambda^{-1}+\lambda\omega^2 f^2=0$, which has the solution

$$\sigma\zeta=\int_0^f\{(1-u)(k+u+u^2)\}^{-\frac{1}{2}}du$$

where

$$\sigma=\int_0^1\{(1-u)(k+u+u^2)\}^{-\frac{1}{2}}du \text{ and } \sigma(\lambda\omega)^{-1}=\int_0^1 u\{(1-u)(k+u+u^2)\}^{-\frac{1}{2}}du$$

to satisfy the boundary conditions; $k=1+3\mu\lambda^{-2}$. When $S(x)$ is given, both μ and x_0 are known. There is only one relation for the two remaining constants λ and ω; this arbitrariness can be removed when intial data for some particular value of x is given.

Upon inversion, the solution for f can be written as

$$f(\zeta)=(2+k)^{\frac{1}{2}}+1 - \frac{2(2+k)^{\frac{1}{2}}}{1+\mathrm{cn}\{(2+k)^{\frac{1}{2}}\sigma(1-\zeta)\}},$$

where cn is the Jacobian Elliptic function with modulus $\{\frac{1}{2}+\frac{3}{4}(2+k)^{-\frac{1}{2}}\}^{\frac{1}{2}}$; see Abramowitz and Stegun [11]. This reduces to the expression gained by Watson when $\mu=0$; there, x_0 is the arbitrary constant.

4. The Momentum Integral Approach

Only the one exact solution to the equations (5), (6) can be found; therefore approximate methods are necessary for further progress. In this section we consider an approach analagous to the momentum integral of Karman where the particular profile is the quartic polynomial introduced by Polhausen (*cf.* Rosenhead [4]). Integration of the momentum equation (6) with respect to y shows

$$\int_0^\eta (uu_x+vu_y)\,dy = S\eta-(u_y)_{y=0}\,.$$

The left-hand side can be simplified upon using (5) together with the conditions (7)–(9) for

$$(u_y)_{y=0} = S\eta - \frac{d}{dx}\int_0^\eta u^2\,dy\,. \tag{17}$$

This is the required momentum integral.

We now assume that, to a certain approximation, we can write $u(x, y)$ in the similarity representation (10). The boundary conditions immediately show that $f(0)=0, f(1)=1, f'(1)=0$. For the second derivatives of $f(\zeta)$, it is seen from (6) that $(u_{yy})_{y=0}=-S$; therefore $f''(0)=-S\eta^2/U$. A similar calculation for the free surface shows $f''(1)=-S\eta^2/U+U'\eta^2$. Higher order derivatives can be found, though when a quartic polynomial is taken to approximate f, these are sufficient.

We write

$$f(\zeta)=\sum_{n=0}^{4} a_n\zeta^n \text{ for } 0\leqq\zeta\leqq 1\,,$$

the coefficients a_n from the five separate conditions given above, for

$$f(\zeta)=2\zeta+\zeta^3+\zeta^4+\tfrac{1}{6}\Lambda\zeta(1-\zeta)^3-\tfrac{1}{6}\Omega\zeta(1+2\zeta)(1-\zeta)^2 \tag{18}$$

where $\Lambda=S\eta^2/U$ and $\Omega=S\eta^2(U-U'\eta)^2$.

The quantities Λ and Ω are assumed to be sufficiently slowly varying functions of x in the usual manner.

The function $f(\zeta)$ is now substituted through (10) into the integrals (9) and (17) to give, after considerable simplification, the differential equations

$$\tfrac{7}{10}U\eta-\tfrac{1}{60}S\eta^3+\tfrac{1}{40}UU'\eta^3=1 \tag{19}$$

and

$$\begin{aligned}(\tfrac{1520}{89}+\tfrac{2}{567}S\eta^3-\tfrac{1504}{945}U^2\eta^2+\tfrac{34}{945}SU\eta^4-\tfrac{1}{1701}S^2\eta^6)\eta'+(\tfrac{1}{567}+\tfrac{34}{2835}U\eta-\tfrac{2}{8505}S\eta^3)S'\eta^4\\ +\frac{1}{U_\eta}(\tfrac{22000}{189}-\tfrac{47024}{189}U\eta+\tfrac{1372}{567}S\eta^3+\tfrac{14192}{135}U^2\eta^2+\tfrac{8152}{2835}SU\eta^4+\tfrac{68}{8505}S^2\eta^6)\,.\end{aligned} \tag{20}$$

These constitute a coupled pair of first order ordinary differential equations for $U(x)$ and $\eta(x)$ in terms of the known function $S(x)$.

One of the basic phenomena of boundary layer theory is that of separation, and if this is to occur in the present context, then the point of separation will be where the drag on the bed of the stream is zero; that is, when $(u_y)_{y=0}=0$. In the present approximation, this occurs when

$$\Lambda-\Omega=U'\eta^2=-12 \tag{21}$$

Separation is brought about when there is a sufficient retardation in the flow, and in this situation it is caused by a reduction in the gravitational forces due to the slope of the incline becoming increasingly horizontal.

In conclusion, we note that when a cubic polynomial is taken instead of the quartic, the elevation $\eta(x)$ is given by a single first order differential equation in terms of $S(x)$; however, this approximation does not include the possibility of separation.

5. Numerical Solutions

The differential equations (19), (20) have been integrated numerically for many different bed profiles $S = S_n(x)$, the solution quoted here are those that most typically represent the pheno-

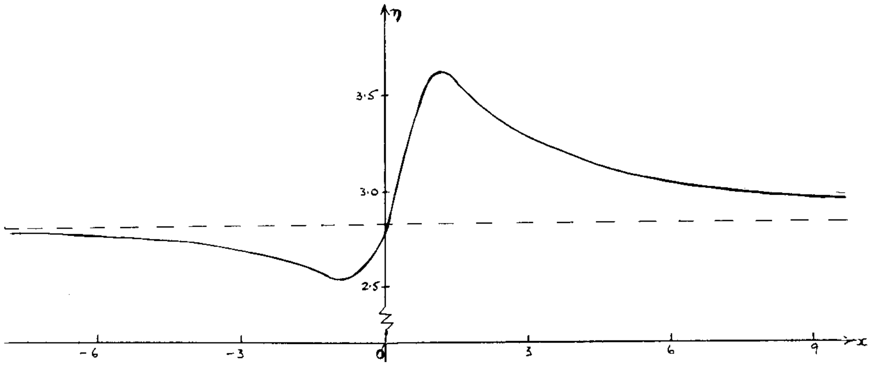

Figure 1. Graph of $\eta(x)$ when $S(x) = \frac{1}{2}\{1 - x(1+x^2)^{-1}\}$.

TABLE 1: $S_2(x) = \frac{1}{2}\{1 - 2x(1+x^2)^{-2}\}$.

Apparent singularity between $x = 0.61568905$ *and* $x = 0.61568911$.

x	U	η	$U'\eta^2$	$U\eta$	$S\eta^2/U$
−5.0	0.6601	2.593	0.0718	1.712	7.049
−4.8	0.6623	2.584	0.0760	1.711	7.051
−4.6	0.6647	2.574	0.0805	1.711	7.054
−4.4	0.6672	2.564	0.0854	1.711	7.057
−4.2	0.6699	2.554	0.0907	1.711	7.060
−4.0	0.6728	2.542	.0964	1.710	7.064
−3.8	0.6759	2.530	0.1026	1.710	7.067
−3.6	0.6792	2.517	0.1093	1.710	7.071
−3.4	0.6828	2.504	0.1166	1.710	7.076
−3.2	0.6866	2.489	0.1244	1.709	7.080
−3.0	0.6908	2.474	0.1327	1.709	7.085
−2.8	0.6953	2.457	0.1415	1.708	7.091
−2.6	0.7002	2.439	0.1505	1.708	7.096
−2.4	0.7054	2.421	0.1596	1.708	7.101
−2.2	0.7111	2.401	0.1681	1.707	7.106
−2.0	0.7171	2.380	0.1752	1.707	7.111
−1.8	0.7234	2.359	0.1794	1.707	7.113
−1.6	0.7299	2.338	−0·1779	1.707	7.111
−1.4	0.7363	2.319	0.1663	1.708	7.103
−1.2	0.7421	2.302	0.1367	1.709	7.084
−1.0	0.7462	2.293	0.0758	1.711	7.045
−0.8	0.7471	2.296	−0.0384	1.715	6.972
−0.6	0.7423	2.323	−0.2415	1.724	6.844
−0.4	0.7279	2.390	−0.5867	1.740	6.63
−0.2	0.6999	2.522	−1.145	1.765	6.293
0.0	0.6555	2.757	−1.999	1.807	5.797
0.2	0.5958	3.146	−3.253	1.874	5.112
0.4	0.5266	3.778	−5.090	1.990	4.207
0.6	0.4564	4.944	−8.173	2.256	3.150

mena. An example that shows the basic features when separation does not occur is given in Figure 1; here $S(x)=\frac{1}{2}\{1-x(1+x^2)^{-1}\}$. At infinity the value of $U\eta$ is approximately 1.714, and the greatest variation from this value throughout the range of integration is about 5%; similarly $S\eta^3 \simeq 12.017$, with a maximum deviation of 14%. The smallest value of $U'\eta^2$ is -1.755, so that there is no suggestion of separation. Nevertheless, there is quite a rapid change in the depth of the liquid slightly upstream of the position of minimum slope, which is at $x=1$ with $(S_1)_{\min}=\frac{1}{4}$.

In the second example to be quoted, we take $S_2(x)=\frac{1}{2}\{1-2x(1+x^2)^{-1}\}$. It is found in this case that there is a breakdown in the numerical procedure at $x \simeq 0.616$ due to an apparent singularity, the values calculated are given in Table 1. Because little difference is observed from the previous, well-behaved, case at the beginning of the numerical integration, the table just gives the results from the value $x=-5$. Mathematically, the only means whereby the equations (19), (20) can develop a singularity is for $\Delta(x)$, the coefficient of η' in (20), to become zero at some particular value of x. This behaviour is noted in the table; $\Delta(\infty) \simeq 4.066$ and Δ decreases very rapidly as the singularity approaches with $\Delta(0.4) \simeq 2.327$ and $\Delta(0.6) \simeq 0.514$. The slope $S_2(0.616) \simeq 0.054$. A very rough analysis of these equations also indicates that $U\eta$ must be greater $\sqrt{5}$ for Δ to be zero, and this in turn certainly requires $U'\eta^2$ to be less than -10 from (19). Consequently, it is not unreasonable to maintain that the singularity occurs at a point of separation where $U'\eta^2=-12$. It has not been felt necessary to pursue these arguments more precisely, no other mechanism seems likely cause this separation and the correlation appears to be sufficiently demonstrated. The difference between the two cases so far presented seems to indicate that it is the reduction of the gravitational force due to the slope becoming increasingly separation to occur.

As further evidence towards this conclusion, the slope $S_3(x)=\frac{1}{2}\{1-\sqrt{6}\,x(1+6x^2)^{-1}\}$ was investigated; there is a more rapid change in the slope, but $(S_3)_{\min}$ is just $\frac{1}{4}$. The range of values of $U\eta$ and $U'\eta^2$ are found to be only a little greater than those present with S_1, and no point of separation is indicated.

When the bed has the form of a circle with radius a, the slope is given by $S(x)=\sin\gamma-(x/a)$ for constant γ. The numerical integration was carried through for $a=6$, $a=2$ (starting at $x=0$ for $\gamma=\frac{1}{6}\pi$). In each case the singularity occurs after a gradual decrease in the values of $U'\eta^2$ towards -12. When the circle has radius 6, the singularity occurs when the slope makes an angle 1° 7′ with the horizontal; when the circle has radius 2, the critical slope is 2° 45′. As could have been anticipated, separation occurs earlier when the slope is more rapidly changing, though the order of magnitude of the two slopes is equivalent.

To check the accuracy of this approximate method, a numerical integration was completed for the exact solution given in section 3; we took $S_4(x)=(x+2)^{-3}$. Once the adjustment to the initial data is complete, it is found that the variation in the quantity $U\eta$ is less than 0.1%, and in $U'\eta^2$ is about 1% when x is taken over the domain $0 \leq x \leq 10$. These variations are well within the limits that can be considered satisfactory.

6. Analytical Approximations

There is just one direction in which information has been found by analytical processes. When the slope of the incline is given by the infinite series

$$S(x) = \sum_{n=0}^{\infty} a_n x^{-n} \quad \text{for some region} \quad -\infty < x < x_1\,,$$

we can write the stream function ψ (defined by $u=\psi_y$, $v=-\psi_x$) in the form

$$\psi(x, y) = \sum_{n=0}^{\infty} \psi_n(y) x^{-n}\,.$$

In particular, $\psi_0=a_0(\frac{1}{2}hy^2-\frac{1}{6}y^3)$ from (4). The resultant free surface elevation is written as

$$\eta(x) = \sum_{n=0}^{\infty} c_n x^{-n} \text{ wigh } c_0 = h\,.$$

To calculate the further functions $\psi_n(y)$, we solve $\psi_y\psi_{xy} - \psi_x\psi_{yy} = S + \psi_{yyy}$ subject to the conditions $\psi = \psi_y = 0$ on $y = 0$; $\psi =$ constant, $\psi_{yy} = 0$ on $y = \eta$. The procedure is straighforward, and the results for $n = 1, 2$ are found to be

$$\psi_1(y) = \tfrac{1}{3}a_1 y^2 (h - \tfrac{1}{2}y) \qquad c_1 = -\tfrac{1}{3}a_1 h/a_0$$

$$\psi_2(y) = \frac{h}{315}(105a_2 - 35a_1^2/a_0 + 4a_0 a_1 h^4)y^2 - \frac{a_2}{6}y^3 - \frac{a_0 a_1}{2520}(14h^2 y^5 - 7hy^6 + y^7)$$

$$c_2 = \frac{a_1^2 h}{9a_0} - \frac{a_2 h}{3a_0} - \frac{2a_1 h^5}{105}\,.$$

These representations do show the possibility of a point of separation, even to the first approximation; however, the numerical calculations indicate that many more terms are required to give a true position with any accuracy.

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